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DISCONTINUOUS SOLUTIONS IN HEATED VISCOUS INCOMPRESSIBLE FLUIDS

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Equations for propagation of surfaces of strong discontinuities in a viscous incompressible heat-conducting fluid are obtained.

We consider the existence of discontinuous classes of motion in a viscous incompressible liquid medium with allowance for temperature stresses based on the general theory of characteristics.

The corresponding nonstationary processes are analyzed based on the vector equation of motion, the continuity equation, the equation of state, and the heat-conduction equation disregarding dissipation of mechanical energy [1, 2]

$$\mu\Delta\mathbf{v} + \left(\eta + \frac{\mu}{3}\right)\text{grad div } \mathbf{v} - \text{grad } p = \rho \frac{d\mathbf{v}}{dt}\,,$$
$$\frac{d\rho}{dt} + \rho \text{ div } \mathbf{v} = 0\,, \quad p = f(\rho, T)\,, \quad q = F(T)\,, \tag{1}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity vector, p is the pressure, ρ is the density, η and μ are the volume (dilatational) and shear viscosities, q is the heat flux, and T is the absolute temperature. In order to simplify system (1) we will assume that the fluid is incompressible (ρ = const) and the motions occur with low velocities. We have

$$\mu\Delta\mathbf{v} - \text{grad } p = \rho \frac{\partial\mathbf{v}}{\partial t}\,, \quad \text{div } \mathbf{v} = 0\,, \quad q = F(T)\,. \tag{2}$$

Let the solution of the system of equations (2) have a strong discontinuity on the surface $\varphi(t, x_1, x_2, x_3)$ = const, i.e., upon passage through this surface the functions p, T, and v_j, $j = \overline{1, 3}$ remain continuous and certain derivatives of first order have a discontinuity of the first kind on this surface. Then, for the partial derivatives $\frac{\partial v_m}{\partial x_n}$, $\frac{\partial v_m}{\partial t}$, $\frac{\partial p}{\partial x_n}$, $\frac{\partial p}{\partial t}$, $\frac{\partial T}{\partial x_n}$, and $\frac{\partial T}{\partial t}$, $m, n = \overline{1, 3}$, we write the kinematic compatibility conditions [3, 4]

$$p_k \frac{\partial v_j}{\partial t} - p_0 \frac{\partial v_j}{\partial x_k} = M_{kj}\,, \tag{3}$$

$$p_k \frac{\partial p}{\partial t} - p_0 \frac{\partial p}{\partial x_k} = M_{k4}\,, \tag{4}$$

$$p_k \frac{\partial T}{\partial t} - p_0 \frac{\partial T}{\partial x_k} = M_{4k}\,, \tag{5}$$

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where M_{kj}, M_{k4}, and M_{4k} are continuous functions and $p_0 = \frac{\partial\varphi}{\partial t}$ and $p_k = \frac{\partial\varphi}{\partial x_k}$, $j,\ k = \overline{1, 3}$. Furthermore, in propagation of the discontinuity surface, the dynamic compatibility conditions must be satisfied, i.e., Eqs. (2) remain true in regions bounded by the surface $\varphi(t, x_1, x_2, x_3) = \text{const}$:

$$\mu\Delta v_j - \operatorname{grad} p - \rho\frac{\partial v_j}{\partial t} = M_j\,, \tag{6}$$

$$\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k} = M_4\,, \tag{7}$$

$$F(T) = M_5\,. \tag{8}$$

Here M_j, $j = \overline{1, 5}$ are also continuous functions. If the system of equations (3)–(8) is solvable, the first partial derivatives will be expressed in terms of continuous functions and will remain continuous themselves. Therefore the discontinuity surface $\varphi = \text{const}$ will be determined from the condition of unsolvability of the system of equations (3)–(8) for these derivatives. In view of the fact that the given system contains twenty equations, it will be appropriate to reduce their number before proceeding to further calculations. Let us do this for different models of the heat-conducting liquid medium.

In the first case we will assume that the thermal processes in the incompressible fluid are described by the classical law of heat conduction [5]

$$\lambda\Delta T = \rho c_p\frac{\partial T}{\partial t}\,, \tag{9}$$

where λ is the thermal-conductivity coefficient and c_p is the specific heat. Then Eq. (8) will acquire the form

$$\lambda\Delta T - \rho c_p\frac{\partial T}{\partial t} = M_5\,. \tag{10}$$

We differentiate Eqs. (3) and (5) with respect to x_l and rewrite the kinematic compatibility conditions as follows:

$$\frac{\partial v_j}{\partial t}\frac{\partial p_k}{\partial x_l} + \ldots = p_0\frac{\partial^2 v_j}{\partial x_l\partial x_k}\,,\quad p_k\frac{\partial p}{\partial t} - M_4 = p_0\frac{\partial p}{\partial x_k}\,,$$

$$\frac{\partial T}{\partial t}\frac{\partial p_k}{\partial x_l} + \ldots = p_0\frac{\partial^2 T}{\partial x_l\partial x_k}\,. \tag{11}$$

Now we multiply Eqs. (6), (7), and (10) by p_0 and replace the resultant products $p_0\frac{\partial^2 v_j}{\partial x_n^2}$, $p_0\frac{\partial p}{\partial x_n}$, and $p_0\frac{\partial^2 T}{\partial x_n^2}$, $j,\ n = \overline{1, 3}$ by the left-hand sides of expressions (11). As a result we obtain a system of five equations for $\frac{\partial v_j}{\partial t}$, $\frac{\partial p}{\partial t}$, and $\frac{\partial T}{\partial t}$, $j = \overline{1, 3}$:

$$\frac{\partial v_j}{\partial x_j}\left(\mu \sum_{k=1}^{3} \frac{\partial p_k}{\partial x_k} - \rho p_0\right) - p_j \frac{\partial p}{\partial t} + \ldots = 0\,, \quad j = \overline{1, 3}\,,$$

$$\sum_{k=1}^{3} p_k \frac{\partial v_k}{\partial t} + \ldots = 0\,, \tag{12}$$

$$\frac{\partial T}{\partial t}\left(\lambda \sum_{k=1}^{3} \frac{\partial p_k}{\partial x_k} - \rho c_p p_0\right) + \ldots = 0\,.$$

The condition of unsolvability of system (12) for the derivatives indicated above makes it possible to write the equation of propagation of strong discontinuities

$$\det \|a_{kl}\| = 0$$

or

$$\begin{Vmatrix} \mu\Delta\varphi - \rho p_0 & 0 & 0 & -p_1 & 0 \\ 0 & \mu\Delta\varphi - \rho p_0 & 0 & -p_2 & 0 \\ 0 & 0 & \mu\Delta\varphi - \rho p_0 & -p_3 & 0 \\ p_1 & p_2 & p_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda\Delta\varphi - \rho c_p p_0 \end{Vmatrix} = 0\,. \tag{13}$$

Expanding the determinant (13), we obtain

$$g^2\,(\mu\Delta\varphi - \rho p_0)^2\,(\lambda\Delta\varphi - \rho c_p p_0) = 0\,. \tag{14}$$

Equation (14) yields the equations of strong discontinuities

$$\mu\Delta\varphi - \rho p_0 = 0\,, \tag{15}$$

$$\lambda\Delta\varphi - \rho c_p p_0 = 0\,, \tag{16}$$

and the existence of the surface of a stationary discontinuity [6]

$$g^2 = 0\,. \tag{17}$$

The equations of strong discontinuities (15) and (16) can be used in obtaining the front of a shock wave and in deriving dispersion relations by means of the corresponding substitutions into these equations. Thus, for plane homogeneous waves we have from (15) and (16) the known dispersion laws for transverse (shear) waves [7]

$$k = \pm\sqrt{\frac{\omega\rho}{2\mu}}\,(1 + i)\,, \quad k = \pm\sqrt{\frac{\rho c_p \omega}{2\lambda}}\,(1 + i)\,. \tag{18}$$

Equations (18) that relate the wave number k to the frequency ω can be used for calculating the phase velocity $v = \dfrac{\omega}{\mathrm{Re}\,k(\omega)}$ and the damping factor $\alpha = \mathrm{Im}\,k(\omega)$ [7].

For a viscous incompressible fluid with a finite velocity of propagation of heat, the law of heat conduction (in the absence of heat sources) has the following form [8]:

$$\lambda \Delta T = \rho c_p \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \right), \tag{19}$$

where τ_0 is the relaxation time of the heat flux. Using (19), we write the dynamic compatibility condition (8) as follows [3, 4]:

$$\sum_{k=1}^{3} \lambda p_k \frac{\partial T}{\partial x_k} - \rho c_p \left(p_0 + \tau_0 p_0 \frac{\partial T}{\partial t} \right) = M_5 \,. \tag{20}$$

We transform the system of equations (6), (7), and (20) according to an analogous scheme. Ultimately, we obtain the following system of equations for the first partial derivatives of p, T, and v_j, $j = \overline{1, 3}$ with respect to time:

$$\frac{\partial v_j}{\partial x_j} \left(\mu \sum_{k=1}^{3} \frac{\partial p_k}{\partial x_k} - \rho p_0 \right) - p_j \frac{\partial p}{\partial t} + \ldots = 0 \,, \quad j = \overline{1, 3} \,,$$

$$\sum_{k=1}^{3} p_k \frac{\partial v_k}{\partial t} + \ldots = 0 \,, \tag{21}$$

$$\frac{\partial T}{\partial t} (\lambda g^2 - \rho c_p \tau_0 p_0^2) + \ldots = 0 \,.$$

Here $g^2 = p_1^2 + p_2^2 + p_3^2$. We obtain the condition of unsolvability of system (21) for $\frac{\partial v_j}{\partial t}$, $\frac{\partial p}{\partial t}$, and $\frac{\partial T}{\partial t}$, $j = \overline{1, 3}$ by setting the determinant composed of the coefficients of these derivatives equal to zero:

$$\det \| A_{kl} \| = 0 \,. \tag{22}$$

It can easily be seen that the determinants (21) and (13) differ only in the expressions $a_{55} = \lambda \Delta \varphi - \rho c_p p_0$ and $A_{55} = \lambda g^2 - \rho c_p \tau_0 p_0^2$. Then from (21) we obtain

$$g^2 (\mu \Delta \varphi - \rho p_0)^2 (\lambda g^2 - \rho c_p \tau_0 p_0^2) = 0$$

or

$$\mu \Delta \varphi - \rho p_0 = 0 \,, \tag{23}$$

$$\lambda g^2 - \rho c_p \tau_0 p_0^2 = 0 \,, \tag{24}$$

$$g^2 = 0 \,. \tag{25}$$

Equations (23) and (25) coincide with Eqs. (15) and (17) in view of the absence of the effect of connectedness of the velocity and temperature fields. Equation (24) makes it possible to calculate the velocity of propagation of a heat wave [3, 4]:

$$v_{\mathrm{h}} = -\frac{p_0}{g} = \sqrt{\frac{\lambda}{\rho c_p \tau_0}}\ .$$

In conclusion we note that the equations of strong discontinuities (15), (16), and (24) coincide exactly with the characteristic surfaces for the considered models of a viscous heat-conducting liquid medium [9].

NOTATION

k, wave number; ω, frequency; Re and Im, real and imaginary parts of the complex number; i, imaginary unit; v_{h}, heat-wave velocity.

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